BSc-part-I(H), Paper-I Abstract Algebra: Subgroups. 18-07-2020

Q.1. If H, K are subgroups of a group G, then HK is a subgroup of G iff $HK = KH$.

Proof: Let H and K subgroups of a group G so that

$$HH^{-1} = H, \quad KK^{-1} = K \longrightarrow (1)$$

~~Step~~ and $K^{-1} = K, H^{-1} = H \longrightarrow (2)$

Step I Let HK be a subgroup of G so that

$$(HK)^{-1} = HK \longrightarrow (3)$$

We have to show that $HK = KH \longrightarrow (4)$

From (3), $K^{-1}H^{-1} = HK$

From (2), We have $HK = KH$

Step II. Let $HK = KH$

We have to show that HK is a subgroup of G.

For, this we have to prove that

$$(HK)(HK)^{-1} = HK \longrightarrow (5)$$

Let us consider $(HK)(HK)^{-1} = (HK)(K^{-1}H^{-1}) = H(KK^{-1})H^{-1}$ [By associativity]

$= HKH^{-1}$ [by (1)]

$= KHH^{-1}$ [by (4)]

$= K(HH^{-1}) = KH = HK$ [by (1)]

Q.2: If H and K are subgroups of ~~proved~~ an abelian group G, then HK is a subgroup of G.

Proof: Let us suppose H and K are subgroups of an abelian group G, so that

$$H^{-1} = H, \quad K^{-1} = K \longrightarrow (1)$$

and $HH^{-1} = H, \; KK^{-1} = K \longrightarrow (2)$

We have to show that (HK) is a subgroup of G, we have to prove that

$$(HK)(HK)^{-1} = HK$$

Now, since G is abelian $\Rightarrow ab = ba, \; \forall a, b \in G$

$\therefore h \in H, k \in K \Rightarrow hk = kh \Rightarrow HK = KH \longrightarrow (3)$

Rest part of Q.2. Proof. (Ajanta)

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Let us consider $(HK)(HK)^{-1} = (HK)(K^{-1}H^{-1}) = H(KK^{-1})H^{-1}$

$= HKH^{-1} = (HK)H^{-1} = (KH)H^{-1}$

$= K(HH^{-1})$

$= KH = HK.$

Q.3. A necessary and sufficient condition for a non-empty finite subset H of a group G, with respect to multiplication to be a subgroup is that H must be closed with respect to multiplication i.e.

$a \in H, b \in H \Rightarrow ab \in H$

Proof:→ Necessary Condition:→ Let us suppose that H is a subgroup of G. Then H must be closed with respect to multiplication. Therefore, $a \in H$, $b \in H \Rightarrow ab \in H$

Sufficient condition:→ Here, it is given that H is closed to multiplication i.e. $a \in H, b \in H \Rightarrow ab \in H$.

⇒ Closure property is satisfied

Associativity. $a(bc) = (ab)c$, $\forall a, b, c \in H$

For all, $a, b, c \in H \Rightarrow a, b, c \in G$

$\Rightarrow a(bc) = (ab)c$ (by associativity in G)

Existence of identity. Let e be the identity in G. By the given condition

$a \in H, a \in H \Rightarrow a^2 \in H \Rightarrow a \cdot a \in H$

$\Rightarrow a^3 = a^2 \cdot a \in H \Rightarrow a^4 = a^3 \cdot a \in H$

Proceeding in this way, we see that all the elements

$a, a^2, a^3, a^4, \ldots, a^r, \ldots, a^s$

belongs to H if $a \in H$.

But H is a finite set. Consequently all these element are not distinct, i.e. there must be repitition is this collection of elements, i.e. $a^r = a^s$, where $r, s \in H$ and $r > s$

$\Rightarrow a^{r-s} = a^0 \Rightarrow a^{r-s} = e$.

Also, $(r-s)$ is the positive integer

$\Rightarrow e = a^{r-s} \in H \Rightarrow e \in H$.

Existence of inverse: $r-s \geq 1 \Rightarrow r-s-1 \geq 0 \Rightarrow a^{r-s-1} \in H$

$\Rightarrow a^{r-s} \cdot a^{-1} \in H \Rightarrow e \cdot a^{-1} \in H$

Thus, $a \in H \Rightarrow a^{-1} \in H, \forall a \in H$.

This shows that every element of H is inversible.

Hence, H is a subgroup of G. proved.